On the length expectation values in quantum Regge calculus

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Abstract

Regge calculus configuration superspace can be embedded into a more general superspace where the length of any edge is defined ambiguously depending on the 4-tetrahedron containing the edge. Moreover, the latter superspace can be extended further so that even edge lengths in each the 4-tetrahedron are not defined, only area tensors of the 2-faces in it are. We make use of our previous result concerning quantisation of the area tensor Regge calculus which gives finite expectation values for areas. Also our result is used showing that quantum measure in the Regge calculus can be uniquely fixed once we know quantum measure on (the space of the functionals on) the superspace of the theory with ambiguously defined edge lengths. We find that in this framework quantisation of the usual Regge calculus is defined up to a parameter. The theory may possess nonzero (of the order of Plank scale) or zero length expectation values depending on whether this parameter is larger or smaller than a certain value. Vanishing length expectation values means that the theory is becoming continuous, here *dynamically* in the originally discrete framework.

In our previous works [1, 2] we have developed the viewpoint on quantisation of the Regge calculus as a particular case of a more general system, the area tensor Regge calculus. In [1] the area tensor Regge calculus has been quantised and shown to lead to finite expectation values of areas. In [2] it has been shown that quantisation of the ordinary Regge calculus is uniquely defined under natural physical assumptions provided that this system is considered as a particular case of the system where the same edge can have different lengths in the different 4-tetrahedra containing it. Quantisation of the latter system is considered to be already defined in the form of a quantum measure.

Now we would like to combine these results to define quantum measure in the Regge calculus proceeding from the area tensor Regge calculus whose quantisation (issued from the canonical approach) is consistently definable. To do this yet one step is required since we must relate area tensor Regge calculus and the above mentioned generalisation of Regge calculus with multiply defined edge lengths. Equivalently to say, the latter system can be viewed as collection of the 4-simplices not necessarily having the same edge lengths on junctions between them; at the same time, a 4-simplex in area tensor Regge calculus does not even possess certain edge lengths or metric, only tensors of the 2-faces.

We begin with the result of [1] according to which the vacuum expectation value of a functional on the set of area tensors π and connection matrices Ω in the Euclidean signature case takes the form

$$\begin{aligned}
< \Psi(\{\pi\},\{\Omega\}) > \; &= \int \Psi(-i\{\pi\},\{\Omega\}) \exp\left(-\sum_{\substack{t-\mathrm{like}\\ \sigma^2}} \tau_{\sigma^2}\circ R_{\sigma^2}(\Omega)\right) \\
& \exp\left(i\sum_{\substack{\mathrm{not}\\ t-\mathrm{like}\\ \sigma^2}} \pi_{\sigma^2}\circ R_{\sigma^2}(\Omega)\right) \prod_{\substack{\mathrm{not}\\ t-\mathrm{like}\\ \sigma^2}} \mathrm{d}^6\pi_{\sigma^2} \prod_{\sigma^3} \mathcal{D}\Omega_{\sigma^3} \\
&\equiv \int \Psi(-i\{\pi\},\{\Omega\}) \mathrm{d}\mu_{\mathrm{area}}(-i\{\pi\},\{\Omega\}). \qquad (1)
\end{aligned}$$

Here $A \circ B \stackrel{\mathrm{def}}{=} \frac{1}{2}A^{ab}B_{ab}$. The field variables are the area tensors π_{σ^2} on the triangles σ^2 (the 2-simplices σ^2) and SO(4) connection matrices Ω_{σ^3} on the tetrahedrons σ^3 (the 3-simplices σ^3). This looks as the usual field-theoretical path integral expression with exception of the following three points. First, occurence of the Haar measure on the group SO(4) of connections on separate 3-faces $\mathcal{D}\Omega_{\sigma^3}$. This is connected with the specific form of the kinetic term $\pi_{\sigma^2} \circ \Omega^\dagger_{\sigma^2}\dot{\Omega}_{\sigma^2}$ which appears when one passes to the continuous limit along any of the coordinate direction chosen as time. Here SO(4) rotation Ω_{σ^2} serves to parameterise limiting form of Ω_{σ^3} when σ^3 tends to σ^2. Such form of the kinetic term in the general 3D discrete gravity model has been deduced by Waelbroeck [3].

Second, there are the terms $\pi \circ R$ in the exponential instead of the Regge action which in the exact connection representation would be the sum of the terms with 'arcsin' of the type $|\pi| \arcsin(\pi \circ R/|\pi|)$ [4]. The reason is that the exponential should be the sum of the constraints times Lagrange multipliers in order to fit the canonical quan-

tisation prescription in the continuous time limit whatever coordinate is taken as a time. These constraints just do not contain 'arcsin' (in empty spacetime; situation is more complicated in the presence of matter fields!).

Third, occurence of a set of the triangles integrations over area tensors τ_{σ^2} of which are absent. It is the set of those triangles the curvature matrices on which are functions, via Bianchi identities, of all the rest curvatures. (Integrations over them might result in the singularities of the type of $[\delta(R-\bar{R})]^2$). In the situations when one defines direction of the coordinate called "time", the t-like triangles turn out to be possible choice of the above set (therefore definition in terms of the Bianchi identities can probably serve as definition of what set of triangles can be chosen as timelike in general case).

Here we imply the situation allowing intuitively evident definition of time direction, namely, a certain regular structure [5] of the Regge manifold considered as that consisting of a sequence of the 3D Regge manifolds $t = const$ of the same structure (of linking different vertices) usually called *the leaves of the foliation* along t. The vertices of the considered 3D leaf will be denoted as i, k, l, ... The i^+ means image in the next-in-t leaf of the vertex i taken at the current moment t. The notation $(A_1A_2\dots A_{n+1})$ means unordered n-simplex with vertices A_1, A_2, ..., A_{n+1} (triangle at $n=2$). The 4-simplices are arranged into the 4-prisms between the neighbouring 3D leaves. Let the 4-prism with bases $(iklm)$ and $(i^+k^+l^+m^+)$ consists of the 4-simplices (ii^+klm), (i^+kk^+lm), $(i^+k^+ll^+m)$, $(i^+k^+l^+mm^+)$.

With these definitions, we can divide the whole set of n-simplices into the three groups, namely, first, those of the type $(ii^+A_1\dots A_{n-1})$ containing the edge (ii^+); second, those of the type $(i_1i_2\dots i_{n+1})$ completely located in the leaf; third, those differing from the second type by occurence of the superscript '+' on some of the vertices i_1, i_2, ..., i_{n+1}, e. g. the tetrahedron (i^+klm). We shall refer to these as to the *t-like*, *leaf* and *diagonal* simplices, respectively. These terms can be considered as the different values of a Regge analog of the *world* index in general relativity (GR). The more usual terms *timelike* and *spacelike* will be reserved for the *local frame* indices a, b, c, ... of the tensors.

Let v be tensor of any triangle, π or τ (π and τ are thus Regge analogs of spacelike and timelike components of v w. r. t. the world index).

Note that v_{σ^2} means tensor taken in the local frame of some σ^4 containing σ^2. For this reason, the more detailed notation could be, e. g., $v_{\sigma^2|\sigma^4}$. If simplices are explicitly defined as collections of vertices, we'll enumerate vertices of σ^4 in the subscript and single out among them by brackets the vertices corresponding to the considered object; e. g. $v_{(ABC)DE}$ is tensor of the triangle (ABC) defined in the frame of the 4-simplex $(ABCDE)$. Further, since the number of the triangles N_2 is larger than the number of the 4-simplices N_4 generally there are more than one area tensors defined in the frame of any given 4-simplex. If, say, $v_{\sigma^2_1}$ and $v_{\sigma^2_2}$ are defined in the same frame, we can form the scalar $v_{\sigma^2_1} \circ v_{\sigma^2_2}$ which together with the ten areas of the simplicial 2-faces leads to the additional conditions which should be imposed in order that these tensors unambiguously define ten simplicial edge lengths. That is, there are points in the configuration space of the area tensor Regge calculus which do not correspond to any flat 4-metric in a given 4-simplex. On the other hand, in the simplicial complex where the length of an edge is allowed to be different in the different 4-simplices containing the edge the area of a

2-face can be also different in the different 4-simplices containing the 2-face. At the same time, in our formulation of area tensor Regge calculus each the 2-face σ^2 possesses the area $|v_{\sigma^2}| \equiv (v_{\sigma^2} \circ v_{\sigma^2})^{1/2}$ as the only scalar corresponding to it. That is, and vice versa, there are also points in the configuration space of the Regge calculus with independent simplicial edge lengths which do not correspond to any point in the configuration space of area tensor Regge calculus. Now, if we would like that a configuration space of one theory be in correspondence with a subset of the configuration space of another one, the most natural would be to extend the configuration space of area tensor Regge calculus by introducing into consideration for each the 2-face σ^2 the tensors $v_{\sigma^2|\sigma^4}$ for *all* the 4-simplices σ^4 containing σ^2.

For the measure, such the extension looks trivial, as simply adding, first, integrations over $\mathrm{d}^6\pi_{\sigma^2|\sigma^4}$ other than $\mathrm{d}^6\pi_{\sigma^2}$. Being applied to the functionals of π_{σ^2} only, these new integrations result simply in an infinite normalisation factor (some intermediate regularisation which confines the limits of integrals by some large although finite areas is implied). Second, also integrations over $\mathrm{d}^6\tau_{\sigma^2|\sigma^4}$ should be inserted. Otherwise, if $\tau_{\sigma^2|\sigma^4}$ were treated as parameters, knowing these parameters would allow to almost completely fix geometry of 3D leaves, i. e. dynamics itself.

A priori nothing prevent us to integrate over $\mathrm{d}^6\tau_{\sigma^2|\sigma^4}$ in the whole range of $\tau_{\sigma^2|\sigma^4}$ as independent variables. At the same time, the result of [1] concerning existence of the well-defined area expectation values was obtained just in the assumption of smallness of the tensors τ as compared to 1 (Plankian unity). Therefore for technical reasons we choose to restrict absolute values of the tensors τ. In what follows, the conditions which say that system is the usual Regge manifold are to be imposed, and in this framework it is sufficient to restrict some 4 scalars connected with τ per vertex. This corresponds to fixing lapse-shift vector in the continuum GR, i. e. to fixing gauge. Nondegenerate such anzats amounts, say, to fixing $|\tau_1|^2$, $|\tau_2|^2$, $|\tau_3|^2$, $\tau_1 \circ \tau_2$ for the 4 tensors τ_i at each vertex σ^0. That is, τ_i are certain 4 functions of the 0-simplex σ^0, and a more detailed notations $\tau_i(\sigma^0) \equiv \tau_{\sigma_i^2(\sigma^0)|\sigma^4(\sigma^0)}$ imply choice of the 4-simplex $\sigma^4(\sigma^0)$ and the 3 triangles $\sigma_i^2(\sigma^0)$ meeting at the t-like edge at the vertex σ^0 and spanning this 4-simplex.

In view of the above consideration, the measure in the extended configuration space of area tensor Regge calculus takes the form

$$\begin{aligned}
\mathrm{d}\mu_{\mathrm{area\,extended}} \quad = \quad & \exp\left(-\sum_{\substack{t-\mathrm{like}\\ \sigma^2}} \tau_{\sigma^2} \circ R_{\sigma^2}(\Omega) + i \sum_{\substack{\mathrm{not}\\ t-\mathrm{like}\\ \sigma^2}} \pi_{\sigma^2} \circ R_{\sigma^2}(\Omega)\right) \\
& \prod_{\sigma^0}\left(\delta(\tau_1(\sigma^0) \circ \tau_2(\sigma^0) - \zeta\varepsilon_1\varepsilon_2) \prod_{i=1}^{3} \delta(\tau_i(\sigma^0) \circ \tau_i(\sigma^0) - \varepsilon_i^2)\right) \\
& \left(\prod_{\sigma^2} \prod_{\sigma^4 \supset \sigma^2} \mathrm{d}^6 v_{\sigma^2|\sigma^4}\right) \prod_{\sigma^3} \mathcal{D}\Omega_{\sigma^3}
\end{aligned} \tag{2}$$

where, remind, v means π or τ; $0 < \varepsilon_i \ll 1$ and -1 $< \zeta < 1$ are parameters.

Thus, we have the system, area tensor Regge calculus quantisation of which is defined in the form of quantum measure (2) and which containes Regge calculus with independent

lengths as a particular case. In turn, the latter system containes the ordinary Regge calculus. As a result, the latter corresponds to a hypersurface $\Gamma_{\rm Regge}$ in the configuration space of area tensor Regge calculus. The quantum measure can be viewed as a linear functional $\mu_{\rm area\,extended}(\Psi)$ on the space of the functionals $\Psi(\{\pi\})$ on the configuration space (for our purposes it is sufficient to restrict ourselves by dependence on the set of area tensors $\{\pi\}$, especially as the connection matrices Ω can be more general quantities than simply rotations connecting the neighbouring simplicial frames and their physical interpretation is difficult). Physical assumption is that we can consider ordinary Regge calculus as a kind of a state of this more general area tensor system. We can associate with this state the functional of the form

$$\Psi(\{\pi\}) = \psi(\{\pi\})\delta_{\rm Regge}(\{\pi\}) \tag{3}$$

where $\delta_{\rm Regge}(\{\pi\})$ is (many-dimensional) δ-function with support on $\Gamma_{\rm Regge}$. The derivatives of $\delta_{\rm Regge}$ have the same support, but these would violate positivity in what follows. To be more presize, delta-function is not a function, but can be viewed as such if regularised. If the measure on such the functionals exists in the limit when this regularisation is removed, this allows to define the quantum measure on $\Gamma_{\rm Regge}$,

$$\mu_{\rm Regge}(\cdot) = \mu_{\rm area\,extended}(\delta_{\rm Regge}(\{\pi\})\ \cdot). \tag{4}$$

The δ-function $\delta_{\rm Regge}$ is equal to the product of the two deltas,

$$\delta_{\rm Regge}(\{\pi\}) = \delta_{\rm cont}(\{\pi\})\delta_{\rm metric}(\{\pi\}). \tag{5}$$

Here $\delta_{\rm metric}(\{\pi\})$ singles out hypersurface in the configuration space of the area tensor Regge calculus corresponding to the system with well-defined (although independent) metrics in the different 4-simplices; the $\delta_{\rm cont}(\{\pi\})$ singles out hypersurface in the configuration space of the Regge calculus with independent lengths (metrics) corresponding to the usual Regge calculus.

The $\delta_{\rm cont}$ has been considered in our previous work [2], and now the question is about $\delta_{\rm metric}$. Here situation is even simpler, because the problem reduces to that for a one 4-simplex and $\delta_{\rm metric}$ is the product over separate 4-simplices. In each the 4-simplex we should write out the constraints which enable area tensors to correspond to certain simplicial metric. Introduce temporarily (locally, in a given 4-simplex) a world index. Denote the vertices of a 4-simplex by 0, 1, 2, 3, 4 so that (04) is t-like edge. Denote tensor of the triangle $(\lambda\mu 4)$ (λ, μ, ν, $\dots = 0, 1, 2, 3$) as $v^{ab}_{\lambda\mu}$. Then conditions which constrain area tensors to be *bivectors* corresponding to certain tetrad of vectors attributed to the four edges (4λ) take the form

$$\epsilon_{abcd}v^{ab}_{\lambda\mu}v^{cd}_{\nu\rho} \sim \epsilon_{\lambda\mu\nu\rho}. \tag{6}$$

In the 36-dimensional configuration minisuperspace of the 6 antisymmetric tensors[1] $v^{ab}_{\lambda\mu}$ the 20 eqs. (6) define the 16-dimensional hypersurface $\gamma(\sigma^4)$. The $\delta_{\rm metric}$ is the product of δ-functions with support on $\gamma(\sigma^4)$ over all the 4-simplices σ^4. The covariant

[1] Also there are the linear constraints of the type $\sum \pm v = 0$ which enable closeness of the 3-faces of our 4-simplex. These constraints are supposed to be already resolved.

form of the constraints (6) w. r. t. the world index means that these δ-functions are scalar densities of certain weight w. r. t. the world index. In the continuum limit, this means that limiting measure also behaves as scalar density at the diffeomorphisms. This meets usual requirements for the continuum measure, but for the weight of this scalar density different values are possible corresponding to the different factors $(\det \|g_{\lambda\mu}\|)^{\alpha}$ in the measure [6, 7]. In Regge calculus, this corresponds to inserting factors of the type of $V^{\eta}_{\sigma^4}$ where V_{σ^4} is the 4-volume and η is a parameter. Thus δ_{metric} takes the form

$$\delta_{\mathrm{metric}} = \prod_{\sigma^4} \int V^{\eta}_{\sigma^4} \delta^{21}(\epsilon_{abcd} v^{ab}_{\lambda\mu|\sigma^4} v^{cd}_{\nu\rho|\sigma^4} - V_{\sigma^4}\epsilon_{\lambda\mu\nu\rho})\, \mathrm{d}V_{\sigma^4}. \tag{7}$$

We can also introduce into consideration the measures which are the prototype of that one suggested by Leutwyler [8] and reproduced by Fradkin and Vilkovisky in the corrected approach [9], $(\det \|g_{\lambda\mu}\|)^{-3/2} g^{00} \mathrm{d}^{10} g_{\lambda\mu}$. For that we should insert $V^2_{\sigma^3}$ into (7) *per vertex* where σ^3 is some leaf 3-face at this vertex. This just leads to $g^{00} \det \|g_{\lambda\mu}\|$ in the continuum limit. However, there are more than one way to attribute 3-face in the leaf to a given vertex. This makes choice of the total factor in the measure highly ambiguous, and each such choice would violate equivalence of the different 3-simplices. Thus, it is the measures of the type of $(\det \|g_{\lambda\mu}\|)^{\alpha} \mathrm{d}^{10} g_{\lambda\mu}$ which have natural Regge analogs[2]. On the other hand, the role of special noninvariant form of the measure $(\det \|g_{\lambda\mu}\|)^{-3/2} g^{00} \mathrm{d}^{10} g_{\lambda\mu}$ was shown in [9] to amount to cancellation of all the (UV) divergences of the type $\delta^{(4)}(0)$ in the effective action which arise in the theory due to it's nonlinearity. These terms arise as coincidence limit of some bilocals. This situation is specific for the continuum and does not take place in the discrete framework; therefore it is reasonable to confine ourselves to the simple scalar density form of the measure.

Above we have described general form of the quantum measure in Regge calculus considered as hypersurface in the superspace of a more general theory. For performing the further estimate it is convenient to make self-antiselfdual decomposition of area tensors, so that v_{σ^2} maps into 3-vectors ${}^{+}\boldsymbol{v}_{\sigma^2}$ and ${}^{-}\boldsymbol{v}_{\sigma^2}$. Locally in the given 4-simplex we introduce for the tensors of the leaf/diagonal $v_{\alpha\beta} \equiv \pi_{\alpha\beta}$ ($\alpha, \beta, \gamma, \ldots = 1, 2, 3$) or t-like $v_{0\alpha} \equiv \tau_{0\alpha}$ triangles the notations

$$\epsilon_{\alpha\beta\gamma} {}^{\pm}\boldsymbol{\pi}^{\gamma} \stackrel{\mathrm{def}}{=} {}^{\pm}\boldsymbol{\pi}_{\alpha\beta}, \qquad {}^{\pm}\boldsymbol{\tau}_{\alpha} \stackrel{\mathrm{def}}{=} {}^{\pm}\boldsymbol{\tau}_{0\alpha}. \tag{8}$$

Then we have

$${}^{\pm}\boldsymbol{\tau}_{\alpha} = \epsilon_{\alpha\beta\gamma} c^{\beta}\, {}^{\pm}\boldsymbol{\pi}^{\gamma} + \frac{1}{2} C \epsilon_{\alpha\beta\gamma}\, {}^{\pm}\boldsymbol{\pi}^{\beta} \times {}^{\pm}\boldsymbol{\pi}^{\gamma} \tag{9}$$

as general solution to the constraints (6). Choose for definiteness selfdual components and suppress index '+'. For the integral $\int (\cdot) \delta_{\mathrm{metric}} \prod_{\sigma^4, \sigma^2 \subset \sigma^4} \mathrm{d}^{36} v_{\sigma^2|\sigma^4}$ we find the product over the 4-simplices of the factors of the type

$$\int (\cdot) C^{\eta-6} [\boldsymbol{\pi}^1 \times \boldsymbol{\pi}^2 \cdot \boldsymbol{\pi}^3]^{\eta-6} \mathrm{d}C \mathrm{d}^3\boldsymbol{c}\, \mathrm{d}^9\boldsymbol{\pi} \mathcal{D}\mathcal{O} \tag{10}$$

[2]Note, however, that, as it is shown below, the situation with backward passing from the constructed measure to the continuum one is more complex than simply getting $(\det \|g_{\lambda\mu}\|)^{\alpha} \mathrm{d}^{10} g_{\lambda\mu}$ in the continuum limit. Rather we obtain this limiting form only if summation in the functional integral is performed over Regge manifolds with a fixed regular structure; if all structures are taken into account, we get different effective exponentials α for the different functionals averaged with the help of this measure.

where $\mathcal{O}$ is an SO(3) rotation which connects selfdual and antiselfdual sectors.

Let us address now the question of convergence of the functional integral and estimate typical area (or length) expectation values. Denote the scale of tensors of the leaf/diagonal triangles π in the given σ^4 by x. Taking into account existence of yet another scale ε introduced when fixing lapse-shift (ε_i in (2)) we see that C, $\boldsymbol{c}$ parameterising $\boldsymbol{\tau}$ by means of the decomposition (9) have the scale x^{-2} and x^{-1}, respectively (more accurately, $C \sim \varepsilon/x^2$ and $\boldsymbol{c} \sim \varepsilon/x$). Integrations (10) can be reduced to that one over $x^{\eta-3}\mathrm{d}x$ and a number of compact ones.

Further, by changing variables a part of the integrations over $\mathcal{D}\Omega_{\sigma^3}$ can be converted to the integrations over $\mathcal{D}R_{\sigma^2}$ with σ^2 running over all the leaf and diagonal 4-simplices. These integrations, as considered in [1], can be factorised at small $\boldsymbol{\tau}$ (corresponding to the choice $\varepsilon \ll 1$). The integrals obtained can be further decomposed into those in the self- and antiselfdual sectors and read

$$\int e^{i\pi \circ R}\mathcal{D}R = \frac{\nu(|\boldsymbol{\pi}|)^2}{|\boldsymbol{\pi}|^4}, \qquad \nu(l) = \frac{2l}{\pi}\int\limits_0^\pi e^{-l/\sin\varphi}\,\mathrm{d}\varphi. \tag{11}$$

Therefore, if tensor of the considered σ^2 is defined in the given σ^4, we have, in addition to the above $x^{\eta-3}\mathrm{d}x$, the factor which behaves as $x^{-2}e^{-\lambda x}$ where λ is a positive bounded from below function of some parameters integrations over which are to be made. For the purposes of studying convergence of expressions defining expectation values we consider simple calculational model replacing $e^{-\lambda x}$ by e^{-x}. Evidently, if more than one area tensor is defined in the frame of σ^4, the corresponding power of $x^{-2}e^{-x}$ should be taken into account as a factor. Consider some regular way of assigning area tensors to the local frames. If $N^{(d)}_k$ is the number of the k-simplices in the d-dimensional Regge manifold then $N^{(3)}_2 = 2N^{(3)}_3$. This means that in the leaf we can attribute to each 3-simplex some two of it's 2-faces. For example, let (ikl) and (ikm) be attributed to $(iklm)$. Define their area tensors in some one of the two 4-simplices meeting at $(iklm)$, say, in the "future" one; let the latter appears to be, say, (ii^+klm). Then we write on some of the vertices i, k, l, m, …in these relations the superscript '+', i. e. shift them to the next time leaf. This defines in a regular way in what 4-simplex the area tensor of each 2-simplex is defined. Namely, (i^+kl), (i^+km) are defined in (i^+kk^+lm); (i^+k^+l), (i^+k^+m) are defined in $(i^+k^+ll^+m)$; no triangles are defined in $(i^+k^+l^+mm^+)$. These are the simplices in the 4-prism with bases $(iklm)$ and $(i^+k^+l^+m^+)$, and in other prisms the area tensors may be chosen to be defined in the same manner, if periodic structure is implied. Of course, the above notations imply that the corresponding simplices exist indeed; a definite one of 24 possible ways of division of the 4-prism into the four 4-simplices is taken. If x is the area scale in the 4-simplex, say, (i^+kk^+lm) in the frame of which the two area tensors are defined, the corresponding factor in the measure for it is

$$x^{\eta-3}\mathrm{d}x \cdot (e^{-x}x^{-2})^2 = e^{-2x}x^{\eta-7}\mathrm{d}x. \tag{12}$$

Next introduce into consideration $\delta_{\rm cont}$ and integrations not performed thus far. The

$\delta_{\rm cont}$ has been found [2] to read symbolically

$$\delta_{\rm cont} = \prod_{\sigma^3} V^4_{\sigma^3} \delta^6(\Delta_{\sigma^3} S_{\sigma^3}) \left(\prod_{\sigma^2} V^3_{\sigma^2} \delta^3(\Delta_{\sigma^2} S_{\sigma^2}) \right)^{-1} \prod_{\sigma^1} V^2_{\sigma^1} \delta(\Delta_{\sigma^1} S_{\sigma^1}). \tag{13}$$

Here S_{σ^k} is the edge component metric [10] on the k-simplex σ^k (simply collection of $k\frac{k+1}{2}$ edge lengths squared). The $\Delta_{\sigma^k} S_{\sigma^k}$ is discontinuity of this metric induced from a certain pair of the different 4-simplices sharing this k-simplex when passing across σ^k from one of these 4-simplices to another one. Vanishing metric discontinuities on the 3-faces are conditions to be imposed on the Regge calculus with independent 4-simplex metrics to get usual Regge calculus. The δ-functions of discontinuities on the 2-faces and 1-faces (links) serves to cancel effect of the cycles enclosing the triangles and leading to the singularities of the type of δ-function squared. Occurrence of the δ-function in the denominator means that the same function is contained in the numerator and is thereby cancelled.

The $\delta_{\rm cont}$ imposes the constraints required to "glue" together different 4-simplex metrics. The constraints on the scalar areas have been discussed in [11, 12]. In our case of area tensors a new possibility arises to get a system of bilinear constraints. This possibility is just connected with used by us extension of the set of area tensors to the frames of all the 4-simplices containing a given triangle. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}^\prime_1$, $\boldsymbol{v}^\prime_2$, $\boldsymbol{v}^\prime_3$ be triads of area vectors (3-vector projections of selfdual parts of area tensors) in the 3-face induced from the two 4-simplices sharing this face. Upon expressing the lengths in terms of area tensors the corresponding δ-factor in $\delta_{\rm cont}$ can be rewritten as

$$[\boldsymbol{v}_1 \times \boldsymbol{v}_2 \cdot \boldsymbol{v}_3]^4 \delta^6(\boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\beta - \boldsymbol{v}^\prime_\alpha \cdot \boldsymbol{v}^\prime_\beta). \tag{14}$$

The overall set of constraints defined by $\delta_{\rm metric}$ and $\delta_{\rm cont}$ turns out to be bilinear w. r. t. the area tensors. These constraints considered in our work [13] single out Regge calculus hypersurface in the space of all arbitrary sets of area tensors, $\{v\}$.

The role of $\delta_{\rm cont}$ in our estimate is, roughly, in equating the scales of area tensors of the leaf and diagonal triangles x_1 and x_2 in each pair of the neighbouring 4-simplices. Besides x, we have the scale of area tensors of the t-like triangles ε, and it is important that at $\epsilon \ll x$ the $\delta_{\rm cont}$ is invariant w. r. t. rescaling ε and, separately, overall rescaling the scales x in the different 4-simplices. To show this, analyse the factor (14) for the 3-faces of the different types, t-like and leaf/diagonal ones.

If the 3-face is leaf or diagonal one then some of $\boldsymbol{v}_\alpha$, $\boldsymbol{v}^\prime_\alpha$ involved in (14) may turn out to be the algebraic sums of those $\boldsymbol{v}_{\sigma^2|\sigma^4}$ which are taken as independent field variables, as mentioned in the footnote after equation (6). For our regular way of constructing the full 4D Regge manifold from the 3D leaves a 4-simplex has the general form (ii^+ABC) with t-like edge (ii^+) and each of other three vertices A, B, C laying either in the current t leaf $(k, l, m, \dots)$ or in the next-in-t leaf $(k^+, l^+, m^+, \dots)$. Take as independent area tensors in the 4-simplex those ones of the 6 triangles containing, e. g., the vertex i; then others (which are future in t) are expressed on using closure relations. Let the 3-face (i^+klm) be shared by the 4-simplices (ii^+klm) and (i^+kk^+lm). Then, in $\delta_{\rm cont}$, we need to compare future in t area tensors in the former 4-simplex (which are sums of independent

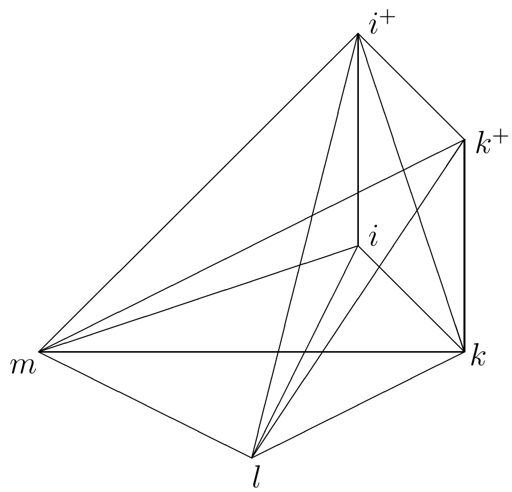

Figure 1: Diagonal 3-face (i^+klm), common for the 4-simplices (ii^+klm) and (i^+kk^+lm).

tensors π and τ in this 4-simplex) with (some combinations of) independent tensors π in the latter 4-simplex. In more detail, we substitute

$$\begin{aligned}
&\boldsymbol{v}_1 = \boldsymbol{\pi}_{(i^+mk)il} = \boldsymbol{\pi}_{(imk)i^+l} + \boldsymbol{\tau}_{(ii^+m)kl} - \boldsymbol{\tau}_{(ii^+k)lm}, \qquad \boldsymbol{v}_1' = \boldsymbol{\pi}_{(i^+mk)k^+l},\\
&\boldsymbol{v}_2 = \boldsymbol{\pi}_{(i^+kl)im} = \boldsymbol{\pi}_{(ikl)i^+m} + \boldsymbol{\tau}_{(ii^+k)lm} - \boldsymbol{\tau}_{(ii^+l)mk}, \qquad \boldsymbol{v}_2' = \boldsymbol{\pi}_{(i^+kl)k^+m},\\
&\boldsymbol{v}_3 = \boldsymbol{\pi}_{(i^+lm)ik} = \boldsymbol{\pi}_{(ilm)i^+k} + \boldsymbol{\tau}_{(ii^+l)mk} - \boldsymbol{\tau}_{(ii^+m)kl}, \qquad \boldsymbol{v}_3' = \boldsymbol{\pi}_{(i^+lm)kk^+} =\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{\pi}_{(klm)i^+k^+} - \boldsymbol{\pi}_{(i^+kl)k^+m} - \boldsymbol{\pi}_{(i^+mk)k^+l}
\end{aligned} \tag{15}$$

in terms of independent variables in this example. Certain sign conventions are implied in the sums. If $\varepsilon \ll x$ then $\boldsymbol{\tau}$ can be neglected as compared to $\boldsymbol{\pi}$ in these sums, and the two scales decouple.

If the 3-face across which metric discontinuity in (14) is taken is t-like, independent area vectors for this face are those for two t-like $\boldsymbol{\tau}$ and one leaf/diagonal $\boldsymbol{\pi}$ 2-faces. For example, for $\sigma^3 = (ii^+kl)$ shared by the 4-simplices (ii^+klm) and (ii^+kln) we take

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{\pi}_{(ikl)i^+m}, &\quad \boldsymbol{v}_1' &= \boldsymbol{\pi}_{(ikl)i^+n},\\
\boldsymbol{v}_2 &= \boldsymbol{\tau}_{(ii^+k)lm}, &\quad \boldsymbol{v}_2' &= \boldsymbol{\tau}_{(ii^+k)ln},\\
\boldsymbol{v}_3 &= \boldsymbol{\tau}_{(ii^+l)km}, &\quad \boldsymbol{v}_3' &= \boldsymbol{\tau}_{(ii^+l)kn}.
\end{aligned} \tag{16}$$

In both cases, those ones of t-like or leaf/diagonal 3-face, corresponding factor in $\delta_{\rm cont}$ turns out to be invariant w. r. t. the overall rescaling x in the different 4-simplices. In our one-dimensional model of estimating this corresponds to factor $x_1\delta(x_1 - x_2)$ for the scales x_1, x_2 on the two neighbouring 4-simplices. Therefore the two measures $f(x_1)\mathrm{d}x_1$ and $f(x_2)\mathrm{d}x_2$ are "glued" together to give

$$x f_1(x) f_2(x) \mathrm{d}x \tag{17}$$

for the overall scale x.

On the whole, in our one-dimensional model of estimating we get the measure $x^{\eta-3}\mathrm{d}x$ per one of $N_4^{(4)}$ 4-simplices and the factor $x^{-2}e^{-x}$ per one of $L_2^{(4)}$ leaf/diagonal simplices; glueing the measures with the help of $\delta_{\rm cont}$ as above we get

$$e^{-L_2^{(4)}x}x^{(\eta-2)N_4^{(4)}-2L_2^{(4)}-1}\mathrm{d}x. \tag{18}$$

For the regular way of constructing the 4D Regge calculus from the 3D leaves $N_4^{(4)} = 4N^{(3)}T$, $L_2^{(4)} = 3N_2^{(3)}T$ where T is the number of the leaves. Besides that, $N_2^{(3)} = 2N_3^{(3)}$. This results in the finite nonzero expectation values for x at $\eta > 5$,

$$< x^j >= \left[\frac{2}{3}(\eta-5)\right]^j \tag{19}$$

(at $N_3^{(3)}T \gg j$). In particular, the length scale $< \sqrt{x} > \sim \sqrt{\eta-5}$. At $\eta \leq 5$ we find $< x^j > = 0$. That is, the functional integral is saturated by infinitely small x at such η. In other words, the system becomes continuum dynamically. We can say that there is transition at $\eta = 5$ between the discrete and continuum phases.

Consider the system in the limit of arbitrarily small angle defects when the corresponding edge lengths vary slowly from vertex to vertex (if Regge manifold possesses regular structure). This situation also takes place if we view the Regge manifold as some triangulation of a certain fixed smooth Riemannian manifold and tend a typical triangulation length to zero, i. e. in the continuum limit. It has been proven [14] that modulo partial use of the equations of motion the area tensor Regge calculus results in the continuum limit in the area-generalised Hilbert-Palatini form of GR such that upon postulating the tetrad form of area tensors we get usual GR. In the functional integral integrations over $\mathcal{D}\Omega$ reduce essentially to $\mathrm{d}^{24}\omega^{ab}_\lambda$ per point, ω^{ab}_λ being certain combinations of generators of Ω. This integration gives $(\det\|g_{\lambda\mu}\|)^{-3}$ *per vertex*. This should be combined with $(\det\|g_{\lambda\mu}\|)^{(\eta-7)/2}\mathrm{d}^{10}g_{\lambda\mu}$ *per 4-simplex* which follows upon integrating out $\delta_{\rm metric}$. From dimensionality arguments and invariance properties of $\delta_{\rm cont}$ w. r. t. the rescaling $g_{\lambda\mu}$ combining separate measures of such form yields

$$(\det\|g_{\lambda\mu}\|)^\alpha \mathrm{d}^{10}g_{\lambda\mu}, \qquad \alpha = \left(\frac{\eta}{2}-1\right)n - \frac{11}{2}. \tag{20}$$

Here $n = N_4^{(4)}/N_0^{(4)}$, the number of the 4-simplices per point. Thus α is not an universal constant for the given theory with certain η. Only if we restrict ourselves to summation in the path integral over Regge manifolds with similar uniform structure, we get a definite α. E. g., for the manifold with fixed simplest periodic structure [15] consisting topologically of hypercubes each composed of 24 4-simplices $n = 24$ which gives $\alpha > 61/2$ in the discrete phase $\eta > 5$ (though, general combinatorial estimate gives $n > 5/2$ which results in weeker bound $\alpha > -7/4$ at $\eta > 5$). In the real case of summation over all the structures we'll find different effective α for the different functionals averaged since the largest contribution will be provided by different structures for the different functionals. Thus, the theory stable at small edge lengths corresponds at large distances in average to rather large α.

To summarize, we have discussed the quantum measure based on the assumption that Regge calculus is a kind of the state of some extended area tensor system with the known quantum measure $d\mu_{\rm area\,extended}$. The measure of interest reads

$$d\mu_{\rm Regge} = \delta_{\rm cont}\delta_{\rm metric}d\mu_{\rm area\,extended}. \tag{21}$$

Here $\delta_{\rm cont}\delta_{\rm metric}$ can be fixed uniquely up to a parameter η in $\delta_{\rm metric}$. For sufficiently large η edge length expectation values are nonzero and finite (of the order of Plank length). In the case of the fixed periodic structure of the Regge manifold the theory at large distances looks as GR with quantum measure $(\det \|g_{\lambda\mu}\|)^\alpha d^{10}g_{\lambda\mu}$ at rather large α, although α depends on this structure and is not an universal constant.

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